CNU- Journal of Higher Education
Volume 17 (2023), p1-14

The SM r-Stirling Numbers: An Algebraic Approach

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Date Submitted: January 22. 2023
Date Revised: April 18, 2023
Originality: 84%
Plagiarism Detection: Passed

ABSTRACT

The r-Stirling numbers by Broder were initially defined through their combinatorial interpretation, and all essential properties and identities were obtained using a combinatorial approach. This paper introduces a slightly modified version of the r-Stirling numbers through their exponential generating functions and derives all necessary properties and identities using an algebraic approach.
Keywords:*Stirling numbers, r-Stirling numbers, generating functions,orthogonality relations,recursive formula, explicit formula, Schlömilch formula*

INTRODUCTION:

The origin of Stirling numbers can be traced back to James Stirling, who introduced them within a purely algebraic framework in his seminal work "Methodus differentialis" (Stirling, 1730). More precisely, the Stirling numbers were introduced as pair of numbers usually denoted by $s(n, k)$ and $S(n, k)$ satisfying the following inverse relations:

$$x^{\underline{n}} = \sum_{k=0}^{n} s(n,k)x^k \qquad (A)$$

$$x^n = \sum_{k=0}^{n} S(n,k)x^{\underline{k}}, \qquad (B)$$

where $x^{\underline{n}} = x(x-1)\dots(x-n+1)$ is called the falling factorial of x of degree n, the values $s(n, k)$ are referred to as Stirling numbers of the first kind, whereas the values $S(n, k)$ are known as Stirling numbers of the second kind.

Throughout the 20th century, numerous mathematicians dedicated their efforts to generalizing and extending this pair of Stirling numbers exploring their applications in combinatorial, probabilistic, and statistical domains. Among these mathematicians, A.Z. Broder (1984) made significant contributions. In Broder's exploration of Stirling numbers, particularly through the lens of permutations and partitions, he crafted a specific generalization known as the r-Stirling numbers. The work of Broder exhibits combinatorial approach in deriving properties and identities of r-Stirling numbers analogous to those of the classical Stirling numbers, unveiling recurrence relations, generating functions, and explicit formulas. Specifically, Broder (1984) defined r-Stirling numbers of the first and second kind as follows

$\begin{bmatrix} n \\ k \end{bmatrix}_r$ = *the number of ways to construct a permutation of the elements in the set* $\{1,\dots,n\}$ *containing k cycles, such that the numbers* $1,2,\dots,r$ *are in different cycles;*

$\begin{Bmatrix} n \\ k \end{Bmatrix}_r$ = *the number of ways to partition the set* $\{1,\dots,n\}$ *into* k *disjoint subsets, such that the numbers* $1,2,\dots,r$ *are in different subsets and each subset must contain at least one element.*

In this present paper, a slightly distinct form of r-Stirling numbers will be introduced by means of their exponential generating functions and necessary properties and identities will be established using algebraic approach. This paper showcases an alternative way of presenting the discussion and results in (Broder, 1984), which can be used as an excellent reference in developing a new variant of Stirling numbers mixing them with the concept of Bernoulli, Euler and Genocchi numbers. This research study not only enriches our understanding of Stirling numbers but also extends their utility across diverse mathematical landscapes.

MATERIALS AND METHODS

Various ways exist for defining a generalized form of Stirling numbers, and the common approach involves introducing them through combinatorial interpretation, recurrence relations or explicit formulas. It is not common to define a generalized form of Stirling numbers by means of their exponential generating functions. However, other special numbers like Genocchi, Bernoulli and Euler numbers and their variations and extensions were usually defined via exponential generating function. Here, we opt to consider a certain generalization of Stirling numbers which are slightly modified version of r-Stirling numbers of Broder (1984) and present these numbers by leveraging their exponential generating function, employing an algebraic approach to derive essential properties and identities. More precisely, the slightly modified r-Stirling numbers find their definition as coefficients within the following exponential generating functions:

$$\sum_{n=0}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{\ln^k(1+t)}{k!} \quad (1)$$

$$\sum_{n=0}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \frac{t^n}{n!} = \frac{e^{rt}(e^t-1)^k}{k!} \quad (2)$$

where $\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r$ and $\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r$ denote the signed first kind r-Stirling numbers and second kind r-Stirling numbers, respectively. Throughout this paper, the term *SM r-Stirling numbers* will be used referring to a slightly modified r-Stirling numbers.

This methodology not only offers a concise representation but also facilitates a deeper understanding of the SM r-Stirling numbers by emphasizing their connection to generating functions. Through this algebraic lens, we unravel the important properties and identities inherent in the r-Stirling numbers of Broder (1984), contributing to a comprehensive exploration of their mathematical nature.

RESULTS AND DISCUSSIONS

In this section, we derive properties and identities of the SM r-Stirling numbers analogous to those properties and identities of Broder's r-Stirling numbers.

Horizontal Generating Functions

The first property to derive is the horizontal generating function. This property is commonly used to define several variations and generalizations of Stirling numbers.

Theorem 1. *The horizontal generating functions for both kinds of SM r-Stirling numbers are given as follows:*

$$(z-r)^{\underline{n}} = \sum_{k=0}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r z^k \quad (3)$$

$$z^n = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r (z-r)^{\underline{k}}. \quad (4)$$

Proof. The exponential generating function in (1) can be written as

$$\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!}\right\} z^k$$

$$= \frac{1}{(1+t)^r}\sum_{k\geq 0} \frac{\ln^k(1+t)}{k!} z^k$$

$$= \frac{1}{(1+t)^r}\sum_{k\geq 0} \frac{[z\ln(1+t)]^k}{k!}$$

$$= \frac{1}{(1+t)^r} e^{\ln(1+t)^z} = (1+t)^z \frac{1}{(1+t)^r}$$

$$= \sum_{n\geq 0} \binom{z-r}{n} t^n = \sum_{n\geq 0} (z-r)^{\underline{n}} \frac{t^n}{n!}.$$

By rewriting the left hand side of the above equation, we get

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r z^k\right\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}(z-r)^{\underline{n}}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields the desired horizontal generating function in (3). On the other hand, the exponential generating function in (2) can be written as

$$\begin{aligned}\sum_{k=0}^{\infty}\left\{\sum_{n=k}^{\infty}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\frac{t^n}{n!}\right\}z^{\underline{k}} &= \sum_{k=0}^{\infty}\left\{\frac{e^{rt}(e^t-1)^k}{k!}\right\}z^{\underline{k}} \\ &= e^{rt}\sum_{k=0}^{\infty}\binom{z}{k}(e^t-1)^k \\ &= e^{rt}\left(1+(e^t-1)\right)^z \\ &= e^{(z+r)t} = \sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}\end{aligned}$$

By rewriting the left hand side of the above equation, we get

$$\sum_{n=0}^{\infty}\left\{\sum_{k=0}^{n}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r z^{\underline{k}}\right\}\frac{t^n}{n!}=\sum_{n=0}^{\infty}(z+r)^n\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ yields (4).

Remark 2. Clearly, when $n<k$, we have

$$\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r=\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r=0.$$

Moreover, when $n=k$,

$$\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r=\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r=1.$$

Remark 3. Replacing z by $-z$ in equation (3) yields

$$(z+r)^{\overline{n}}=\sum_{k=0}^{n}(-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r z^k,$$

where $w^{\overline{n}}=w(w+1)\dots(w+n-1)$ is the rising factorial of w of degree n. This indicates that the first kind r-Stirling numbers of Broder (1984) can be represented using the signed r-Stirling numbers of the first kind

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r=(-1)^{n-k}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r.$$

Orthogonality of SM r-Stirling Numbers

One of the important properties of Stirling-type numbers is the orthogonality relation. This property provides remarkable consequences such as the inverse relation and matrix relation. The following theorem contains the orthogonality relations for both kinds of SM r-Stirling numbers.

Theorem 4. *The orthogonality relation satisfied by the first and second kinds SM r-Stirling numbers is as follows:*

$$\begin{aligned}\sum_{k=0}^{d}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r &= \sum_{k=0}^{d}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r\begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r=\delta_{nm} \\ &= \begin{cases} 0, m\neq n \\ 1, m=n \end{cases}. \qquad (5)\end{aligned}$$

where δ_{nm} is the Kronecker delta and $d\geq n$..

Proof. It's worth noting that equation (3) can be expressed as

$$(z-r)^{\underline{k}}=\sum_{m=0}^{k}\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r z^m.$$

Substituting this to (4) gives

$$\begin{aligned}z^n &= \sum_{k=0}^{n}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\sum_{m=0}^{k}\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r z^m \\ &= \sum_{k=0}^{n}\sum_{m=0}^{k}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r z^m \\ &= \sum_{m=0}^{n}\left\{\sum_{k=m}^{n}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r\right\}z^m.\end{aligned}$$

Hence,

$$\sum_{k=m}^{n}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r=\delta_{nm}=\begin{cases} 0, m\neq n \\ 1, m=n \end{cases}.$$

Similarly, (4) can be written as

$$z^k=\sum_{m=0}^{k}\begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r(z-r)^{\underline{m}}.$$

Substituting this to (3) yields

$$\begin{aligned}(z-r)^{\underline{n}} &= \sum_{k=0}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \sum_{m=0}^{k} \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r (z-r)^{\underline{m}} \\ &= \sum_{k=0}^{n} \sum_{m=0}^{k} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r (z-r)^{\underline{m}} \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \right\} (z-r)^{\underline{m}}.\end{aligned}$$

Hence,

$$\sum_{k=m}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r = \delta_{nm} = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}.$$

Furthermore, since $\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r = \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r = 0$ for $n < k$, we have

$$\begin{aligned}&\sum_{k=0}^{d} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r \\ &= \sum_{k=0}^{d} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \\ &= \delta_{nm} = \begin{cases} 0, m \neq n \\ 1, m = n \end{cases}.\end{aligned}$$

for some $d \geq n$.

Remark 5. Using Remark 3, we have

$$\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r = (-1)^{n-k} \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r.$$

Then, Theorem 4 gives

$$\begin{aligned}&\sum_{k=m}^{n} (-1)^k \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \begin{bmatrix} k+r \\ m+r \end{bmatrix}_r \\ &= \sum_{k=m}^{n} (-1)^k \begin{bmatrix} n+r \\ k+r \end{bmatrix}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \\ &= (-1)^n \delta_{nm} = \begin{cases} 0, m \neq n \\ 1, m = n' \end{cases}\end{aligned}$$

which is exactly the orthogonality relation for both kinds of Broder's r-Stirling numbers.

Remark 6. The following matrix relation is a direct consequence of the orthogonality relation in Theorem 4:

$$\left(\begin{Bmatrix} i+r \\ j+r \end{Bmatrix}_r\right)_n \left(\widehat{\begin{bmatrix} i+r \\ j+r \end{bmatrix}}_r\right)_n = \left(\widehat{\begin{bmatrix} i+r \\ j+r \end{bmatrix}}_r\right)_n \left(\begin{Bmatrix} i+r \\ j+r \end{Bmatrix}_r\right)_n = I_n.$$

Inverse Relations of SM r-Stirling Numbers

Another important property that a special number needs to possess is the inverse relation. This can help transform some generating functions into other forms of relations that have significant meaning in combinatorics.

Theorem 7. *The inverse relation satisfied by the SM r-Stirling numbers is as follows:*

$$f_n = \sum_{k=0}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r g_k \Leftrightarrow g_n = \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r f_k \quad (6)$$

$$f_k = \sum_{n=k}^{\infty} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r g_n \Leftrightarrow g_k = \sum_{n=k}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r f_n. \quad (7)$$

Proof: Using the hypothesis in (4), we have

$$\begin{aligned}\sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r f_k &= \sum_{k=0}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \sum_{m=0}^{k} \widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r g_m \\ &= \sum_{k=0}^{n} \sum_{m=0}^{k} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r g_m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r \right\} g_m \\ &= \sum_{m=0}^{n} \delta_{nm}\, g_m = \delta_{nn} g_n = g_n.\end{aligned}$$

Conversely, we have

$$\begin{aligned}\sum_{k=0}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r g_k &= \sum_{k=0}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \sum_{m=0}^{k} \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r f_m \\ &= \sum_{k=0}^{n} \sum_{m=0}^{k} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r f_m \\ &= \sum_{m=0}^{n} \left\{ \sum_{k=m}^{n} \widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \begin{Bmatrix} k+r \\ m+r \end{Bmatrix}_r \right\} f_m \\ &= \sum_{m=0}^{n} \delta_{nm} f_m = \delta_{nn} f_n = f_n.\end{aligned}$$

To prove the inverse relation in (7), we use

$$\begin{aligned}\sum_{n=k}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r f_n &= \sum_{n=k}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \sum_{m=n}^{\infty} \widehat{\begin{bmatrix} m+r \\ n+r \end{bmatrix}}_r g_m \\ &= \sum_{n=k}^{\infty} \sum_{m=n}^{\infty} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \widehat{\begin{bmatrix} m+r \\ n+r \end{bmatrix}}_r g_m \\ &= \sum_{m=0}^{\infty} \left\{ \sum_{n=0}^{m} \begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r \widehat{\begin{bmatrix} m+r \\ n+r \end{bmatrix}}_r \right\} g_m \\ &= \sum_{m=0}^{\infty} \delta_{mk} g_m = \delta_{kk} g_k = g_k.\end{aligned}$$

Conversely, we have

$$\begin{aligned}\sum_{n=k}^{\infty}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r g_n &= \sum_{n=k}^{\infty}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r \sum_{m=n}^{\infty}\begin{Bmatrix}m+r\\n+r\end{Bmatrix}_r f_m\\ &= \sum_{n=k}^{\infty}\sum_{m=n}^{\infty}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r \begin{Bmatrix}m+r\\n+r\end{Bmatrix}_r f_m\\ &= \sum_{m=0}^{\infty}\left\{\sum_{n=0}^{m}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r \begin{Bmatrix}m+r\\n+r\end{Bmatrix}_r\right\} f_m\\ &= \sum_{m=0}^{\infty}\delta_{mk}f_m = \delta_{kk}f_k = f_k \ . \blacksquare.\end{aligned}$$

Remark 8. Applying the inverse relation in (7) to the generating functions in (1) and (2) gives

$$\begin{aligned}\frac{(1+t)^r t^k}{k!} &= \sum_{n=k}^{\infty}\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r \frac{\ln^n(1+t)}{n!}\\ \frac{t^k}{e^{rt}k!} &= \sum_{n=k}^{\infty}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r \frac{(e^t-1)^n}{n!}.\end{aligned}$$

These are new identities for SM r-Stirling numbers of both kinds.

Triangular Recurrence Relation

The next property to consider is the recurrence relations of both kinds of SM r-Stirling numbers. This relation aids in swiftly computing the initial values of the SM r-Stirling numbers.

Theorem 9. *The SM r-Stirling numbers satisfy the following recursive formulas:*

$$\begin{bmatrix}\widehat{n+r+1}\\k+r\end{bmatrix}_r = \begin{bmatrix}\widehat{n+r}\\k+r-1\end{bmatrix}_r - (r+n)\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r$$

$$\begin{Bmatrix}n+r+1\\k+r\end{Bmatrix}_r = \begin{Bmatrix}n+r\\k+r-1\end{Bmatrix}_r + (k+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r.$$

Proof. In accordance with (3), we may write

$$\begin{aligned}\sum_{k=0}^{n+1}\begin{bmatrix}\widehat{n+r+1}\\k+r\end{bmatrix}_r z^k &= (z-r)^{\underline{n+1}}\\ &= (z-r)^{\underline{n}}(z-r-n)\\ &= (z-r-n)\sum_{k=0}^{n}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r z^k\\ &= \big(z+(-r-n)\big)\sum_{k=0}^{n}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r z^k\\ &= \sum_{k=0}^{n}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r z^{k+1} + (-r-n)\sum_{k=0}^{n}\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r z^k\\ &= \sum_{k=1}^{n+1}\begin{bmatrix}\widehat{n+r}\\k+r-1\end{bmatrix}_r z^k + \sum_{k=0}^{n+1}(-r-n)\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r z^k\\ &= \sum_{k=0}^{n+1}\left\{\begin{bmatrix}\widehat{n+r}\\k+r-1\end{bmatrix}_r + (-r-n)\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r\right\} z^k.\end{aligned}$$

By comparing the coefficients of z^k, we obtain the triangular recursive formula for the signed first kind r-Stirling numbers:

$$\begin{aligned}\begin{bmatrix}\widehat{n+r+1}\\k+r\end{bmatrix}_r &= \begin{bmatrix}\widehat{n+r}\\k+r-1\end{bmatrix}_r\\ &\quad + (-r-n)\begin{bmatrix}\widehat{n+r}\\k+r\end{bmatrix}_r.\end{aligned}$$

Similarly, (4) may be written as

$$\begin{aligned}\sum_{k=0}^{n+1}\begin{Bmatrix}n+r+1\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}} &= z^{n+1} = z^n z\\ &= \sum_{k=0}^{n}\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\, z\\ &= \sum_{k=0}^{n}(z-k+k-r+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &= \sum_{k=0}^{n}(z-r-k)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &\quad + \sum_{k=0}^{n}(k+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &= \sum_{k=0}^{n}\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k+1}}\\ &\quad + \sum_{k=0}^{n}(k+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &= \sum_{k=0}^{n+1}\begin{Bmatrix}n+r\\k+r-1\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &\quad + \sum_{k=0}^{n+1}(k+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r (z-r)^{\underline{k}}\\ &= \sum_{k=0}^{n+1}\left\{\begin{Bmatrix}n+r\\k+r-1\end{Bmatrix}_r + (k+r)\begin{Bmatrix}n+r\\k+r\end{Bmatrix}_r\right\}(z-r)^{\underline{k}}.\end{aligned}$$

Now identifying the coefficients of $(z-r)^{\underline{k}}$, we obtain the following triangular recurrence relation for $\left\{ {n+r \atop k+r} \right\}_r$

$$\left\{ {n+r+1 \atop k+r} \right\}_r = \left\{ {n+r \atop k+r-1} \right\}_r + (k+r)\left\{ {n+r \atop k+r} \right\}_r. \blacksquare$$

Remark 10. Using Remark 3,

$$\begin{aligned}
(-1)^{n+1-k}\left[{\widehat{n+r+1} \atop k+r} \right]_r &= (-1)^{n+1-k}\left[{\widehat{n+r} \atop k+r-1} \right]_r \\
&\quad +(-1)^{n+1-k}(-1)(r+n)\left[{\widehat{n+r} \atop k+r} \right]_r \\
&= (-1)^{n+1-k}\left[{\widehat{n+r} \atop k+r-1} \right]_r \\
&\quad +(r+n)(-1)^{n-k}\left[{\widehat{n+r} \atop k+r} \right]_r \\
&= \left[{n+r \atop k+r-1} \right]_r + (n+r)\left[{n+r \atop k+r} \right]_r,
\end{aligned}$$

which is exactly the triangular recursive formula of Broder's first kind r-Stirling numbers.

To demonstrate the utility of the triangular recurrence relations outlined in Theorem 9, we generate specific values of SM r-Stirling numbers. For the first kind, this is facilitated by the following triangular recurrence relation:

$$\left[{\widehat{n+r+1} \atop k+r} \right]_r = \left[{\widehat{n+r} \atop k+r-1} \right]_r - (r+n)\left[{\widehat{n+r} \atop k+r} \right]_r,$$

with $n=2, k=2, r=2$, we have

$$\left[{\widehat{2} \atop 1} \right]_2 = 0$$

$$\left[{\widehat{3} \atop 2} \right]_2 = \left[{\widehat{2} \atop 1} \right]_2 - 2\left[{\widehat{2} \atop 2} \right]_2 = 0 - 2(1) = -2$$

$$\left[{\widehat{4} \atop 3} \right]_2 = \left[{\widehat{3} \atop 2} \right]_2 - 3\left[{\widehat{3} \atop 3} \right]_2 = -2 - 3(1) = -5$$

$$\left[{\widehat{5} \atop 4} \right]_2 = \left[{\widehat{4} \atop 3} \right]_2 - 4\left[{\widehat{4} \atop 4} \right]_2 = -5 - 4(1) = -9.$$

Now, with $n=3, k=3, r=3$, we have

$$\left[{\widehat{3} \atop 2} \right]_3 = 0$$

$$\left[{\widehat{4} \atop 3} \right]_3 = \left[{\widehat{3} \atop 2} \right]_3 - 3\left[{\widehat{3} \atop 3} \right]_3 = 0 - 3(1) = -3$$

$$\left[{\widehat{5} \atop 4} \right]_3 = \left[{\widehat{4} \atop 3} \right]_3 - 4\left[{\widehat{4} \atop 4} \right]_3 = -3 - 4 = -7$$

$$\left[{\widehat{6} \atop 5} \right]_3 = \left[{\widehat{5} \atop 4} \right]_3 - 5\left[{\widehat{5} \atop 5} \right]_3 = -7 - 5 = -12$$

$$\left[{\widehat{7} \atop 6} \right]_3 = \left[{\widehat{6} \atop 5} \right]_3 - 6\left[{\widehat{6} \atop 6} \right]_3 = -12 - 6 = -18$$

For the second kind SM r-Stirling numbers which is given by this triangular recurrence relation,

$$\left\{ {n+r+1 \atop k+r} \right\}_r = \left\{ {n+r \atop k+r-1} \right\}_r + (k+r)\left\{ {n+r \atop k+r} \right\}_r,$$

with $n=2, k=2, r=2$, we have

$$\left\{ {2 \atop 1} \right\}_2 = 0$$

$$\left\{ {3 \atop 2} \right\}_2 = \left\{ {2 \atop 1} \right\}_2 + (2)\left\{ {2 \atop 2} \right\}_2 = 2$$

$$\left\{ {4 \atop 3} \right\}_2 = \left\{ {3 \atop 2} \right\}_2 + (3)\left\{ {3 \atop 3} \right\}_2 = 5$$

$$\left\{ {5 \atop 4} \right\}_2 = \left\{ {4 \atop 3} \right\}_2 + (4)\left\{ {4 \atop 4} \right\}_2 = 9.$$

With $n=3, k=3, r=3$, we have

$$\left\{ {3 \atop 2} \right\}_3 = 0$$

$$\left\{ {4 \atop 3} \right\}_3 = \left\{ {3 \atop 2} \right\}_3 + (3)\left\{ {3 \atop 3} \right\}_3 = 3$$

$$\left\{ {5 \atop 4} \right\}_3 = \left\{ {4 \atop 3} \right\}_3 + (4)\left\{ {4 \atop 4} \right\}_3 = 7$$

$$\left\{ {6 \atop 5} \right\}_3 = \left\{ {5 \atop 4} \right\}_3 + (5)\left\{ {5 \atop 5} \right\}_3 = 12$$

$$\left\{ {7 \atop 6} \right\}_3 = \left\{ {6 \atop 5} \right\}_3 + (6)\left\{ {6 \atop 6} \right\}_3 = 18.$$

Indeed, the first values of SM r-Stirling numbers can be computed quickly using the triangular recurrence relations.

Explicit Formulas

Another important property of a special number is its explicit formula. This is useful in computing directly the value of the special number for a given specific value of the parameters involved. The subsequent theorem provides the explicit formula for the r-Stirling numbers of the second kind.

Theorem 11. *The formula for the second kind r-Stirling numbers is explicitly stated as follows:*

$$\left\{ {n+r \atop k+r} \right\}_r = \frac{1}{k!}\sum_{i=0}^{k}(-1)^{k-i}\binom{k}{i}(i+r)^n. \quad (8)$$

Proof. Using (2), we obtain

$$\sum_{n\geq k}^{n} k!\left\{ {n+r \atop k+r} \right\}_r \frac{t^n}{n!} = e^{rt}(e^t-1)^k$$
$$= \sum_{i=0}^{k} e^{rt}\binom{k}{i}(e^t)^{k-i}(-1)^i$$
$$= \sum_{i=0}^{k}\binom{k}{i} e^{rt+(k-i)t}(-1)^i$$
$$= \sum_{i=0}^{k}\binom{k}{i} e^{t(r+(k-i))}(-1)^i$$
$$= \sum_{i=0}^{k}\binom{k}{i}\left\{\sum_{n\geq 0}\frac{\left(((k-i)+r)t\right)^n}{n!}\right\}(-1)^i$$
$$= \sum_{n\geq 0}^{k}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left((k-i)+r\right)^n\right\}\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}^{k}\left\{\sum_{i=0}^{k}(-1)^i\binom{k}{i}\left((k-i)+r\right)^n\right\}\frac{t^n}{n!}.$$

Comparing the coefficients of $\frac{t^n}{n!}$ both sides yields

$$k!\left\{ {n+r \atop k+r} \right\}_r = \sum_{i=0}^{k}(-1)^i\binom{k}{i}\left((k-i)+r\right)^n.$$

Replacing i with $k-i$ gives the desired explicit formula in (8). ■

The following theorem provides the explicit formula for the signed first kind r-Stirling numbers. This formula is also recognized as the *Schlömilch-type formula*.

Theorem 12. *The signed first kind r-Stirling number is explicitly defined by the following formula:*

$$\widehat{\left[{n+r \atop k+r} \right]}_r = \sum_{m=k}^{n}\sum_{r=0}^{m-k}\sum_{j=i}^{r}(-1)^{n-m+j+r}\binom{n}{m}\binom{r}{j}\times$$
$$\times\binom{m-1+r}{m-k+r}\binom{2m-k}{m-k+r}\frac{(r-j)^{m-k+r}}{r!}r^{\overline{n-m}}.$$

Proof. When $r=0$, the exponential generating function for the first kind signed r-Stirling numbers reduces to

$$\sum_{n\geq 0} s(n,k)\frac{t^n}{n!} = \frac{1}{k!}[\ln(1+t)]^k$$

where $s(n,k)$ denotes the Stirling numbers of the first kind. Note that the EGF of signed r-Stirling numbers of the first kind which is given by

$$\sum_{k\geq 0}^{n}\widehat{\left[{n+r \atop k+r} \right]}_r \frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r\frac{[\ln(1+t)]^k}{k!}.$$

is composed of two functions. The first function can be expressed as

$$\left(\frac{1}{1+t}\right)^r = (1+t)^{-r}$$
$$= \sum_{n\geq 0}\binom{-r}{n}t^n$$
$$= \binom{-r}{0}t^0 + \sum_{n>0}\binom{-r}{n}t^n$$

where $\binom{-r}{n}$ is the Newton's generalized binomial coefficients. Applying the Newton's Binomial Theorem yields

$$\left(\frac{1}{1+t}\right)^r$$
$$= 1 + \sum_{n>0}\frac{(-r)(-r-1)\dots(-r-n+1)}{n!}t^n$$
$$= 1 + \sum_{n>0}\frac{(-1)^n(r)(r+1)\dots(r+n-1)t^n}{n!}$$
$$= 1 + \sum_{n>0}(-1)^n(r)(r+1)\dots(r+n-1)\frac{t^n}{n!}.$$

And, by the definition of the rising factorial,

$$\left(\frac{1}{1+t}\right)^r = 1 + \sum_{n>0}(-1)^n r^{\bar{n}}\frac{t^n}{n!}$$
$$= \sum_{n\geq 0}(-1)^n r^{\bar{n}}\frac{t^n}{n!}.$$

The second function $\frac{1}{k!}[\ln(1+t)]^k$ can be expressed as

$$\frac{1}{k!}[\ln(1+t)]^k = \sum_{n\geq k} s(n,k)\frac{t^n}{n!}.$$

Hence, using Cauchy's Rule for the product of two power series, we have

$$\sum_{k\geq 0}^{n}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!} = \left(\sum_{n\geq 0}(-1)^n r^{\bar{n}}\frac{t^n}{n!}\right)\left(\sum_{n\geq k}s(n,k)\frac{t^n}{n!}\right)$$

$$= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{r^{\overline{n-m}}t^{n-m}}{(n-m)!}(-1)^{n-m}s(m,k)\frac{t^m}{m!}\right\}$$

$$= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{t^{n-m+m}}{(n-m)!\,m!}(-1)^{n-m}r^{\overline{n-m}}s(m,k)\right\}$$

$$= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{1}{(n-m)!\,m!}(-1)^{n-m}s(m,k)r^{\overline{n-m}}\right\}t^n.$$

Multiplying the summand of the left hand side of the equation by $\frac{n!}{n!}$ yields

$$\sum_{k\geq 0}^{n}\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r \frac{t^n}{n!}$$

$$= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\frac{1}{(n-m)!\,m!}(-1)^{n-m}s(m,k)r^{\overline{n-m}}\right\}\frac{n!}{n!}t^n$$

$$= \sum_{n\geq 0}\left\{\sum_{m=k}^{n}\binom{n}{m}(-1)^{n-m}s(m,k)r^{\overline{n-m}}\right\}\frac{t^n}{n!}.$$

By comparing the coefficients of $\frac{t^n}{n!}$, we have

$$\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r = \sum_{m=k}^{n}\binom{n}{m}(-1)^{n-m}s(m,k)r^{\overline{n-m}}.$$

Using the Schlömilch formula for the Stirling numbers of the first kind

$$s(n,k) = \sum_{r=0}^{n-k}\sum_{j=i}^{r}(-1)^{j+r}\binom{r}{j}\binom{n-1+r}{n-k+r}\binom{2n-k}{n-k+r}\frac{(r-j)^{n-k+r}}{r!},$$

the Schlömilch-type formula for the signed r-Stirling of the first kind is given by

$$\widehat{\begin{bmatrix} n+r \\ k+r \end{bmatrix}}_r = \sum_{m=k}^{n}\binom{n}{m}(-1)^{n-m}\sum_{r=0}^{m-k}\sum_{j=i}^{r}(-1)^{j+r}\binom{r}{j}\binom{m-1+r}{m-k+r}\binom{2m-k}{m-k+r}\frac{(r-j)^{m-k+r}}{r!}r^{\overline{n-m}}$$

$$= \sum_{m=k}^{n}\sum_{r=0}^{m-k}\sum_{j=i}^{r}(-1)^{n-m+j+r}\binom{n}{m}\binom{r}{j}\binom{m-1+r}{m-k+r}\binom{2m-k}{m-k+r}\frac{(r-j)^{m-k+r}}{r!}r^{\overline{n-m}}. \blacksquare$$

Remark 13. Using Remark 3, we have

$$\widehat{\begin{bmatrix} k+r \\ m+r \end{bmatrix}}_r = (-1)^{n-k}\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r$$

$$= \sum_{m=k}^{n}\sum_{r=0}^{m-k}\sum_{j=i}^{r}(-1)^{n-m+j+r}\binom{n}{m}\binom{r}{j}\binom{m-1+r}{m-k+r}\binom{2m-k}{m-k+r}\frac{(r-j)^{m-k+r}}{r!}r^{\overline{n-m}}.$$

$$\begin{bmatrix} n+r \\ k+r \end{bmatrix}_r$$

$$= \sum_{m=k}^{n}\sum_{r=0}^{m-k}\sum_{j=i}^{r}(-1)^{k-m+j+r}\binom{n}{m}\binom{r}{j}\binom{m-1+r}{m-k+r}\binom{2m-k}{m-k+r}\frac{(r-j)^{m-k+r}}{r!}r^{\overline{n-m}}.$$

This is exactly the Schlömilch-type formula obtained in (Corcino et al., 2014) for the r-Stirling numbers of the first kind.

Rational Generating Function

The subsequent theorem includes the rational generating function for the second kind r-Stirling numbers.

Theorem 13. *The rational generating function holds true for the r-Stirling numbers of the second kind, where n and k are non-negative integers:*

$$\psi_k(t) = \sum_{n\geq k}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r t^n = \frac{t^k}{\prod_{j=0}^{k}(1-t(j+r))}.$$

Proof: When $k=0$, we have

$$\psi_0(t) = \sum_{n\geq 0}\begin{Bmatrix} n+r \\ r \end{Bmatrix}_r t^n = \sum_{n\geq 0}r^{n+r-r}t^n$$

$$= \sum_{n\geq 0}r^n t^n = \sum_{n\geq 0}(rt)^n = \frac{1}{1-rt}.$$

Using the triangular recurrence relation in Theorem 9, we get

$$\psi_k(t) = \sum_{n\geq k}\begin{Bmatrix} n+r \\ k+r \end{Bmatrix}_r t^n$$

$$= \sum_{n\geq k}\left[\begin{Bmatrix} n-1+r \\ k+r-1 \end{Bmatrix}_r + (k+r)\begin{Bmatrix} n-1+r \\ k+r \end{Bmatrix}_r\right]t^n$$

$$= \sum_{n\geq k} t \left\{ {(n-1)+r \atop k+r-1} \right\}_r t^{n-1} + t(k+r) \sum_{n\geq k+1} \left\{ {(n-1)+r \atop k+r} \right\}_r t^{n-1}$$

$$= \sum_{n\geq k-1} t \left\{ {(n-1+1)+r \atop k+r-1} \right\}_r t^{n-1+1} + t(k+r) \sum_{n\geq k} \left\{ {(n-1+1)+r \atop k+r} \right\}_r t^{n-1+1}$$

$$= t \sum_{n\geq k-1} \left\{ {n+r \atop k+r-1} \right\}_r t^n + t(k+r) \sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r t^n.$$

Hence,

$$\psi_k(t) = t\psi_{k-1}(t) + t(k+r)\psi_k(t)$$

$$\psi_k(t) = \frac{t}{\big(1-t(k+r)\big)} \psi_{k-1}(t).$$

Thus,

$$\psi_k(t) = \frac{t}{\big(1-t(k+r)\big)} \psi_{k-1}(t)$$

$$= \frac{t}{\big(1-t(k+r)\big)} \frac{t}{\big(1-t((k-1)+r)\big)} \psi_{k-2}(t)$$

$$= \frac{t}{\big(1-t(k+r)\big)} \frac{t}{\big(1-t((k-1)+r)\big)} \cdots \frac{t}{\big(1-t((k-(k-1))+r)\big)} \psi_{k-k}(t)$$

$$= \frac{t}{\big(1-t(k+r)\big)} \frac{t}{\big(1-t((k-1)+r)\big)} \cdots \frac{t}{\big(1-t(1+r)\big)} \frac{1}{1-rt}$$

$$\psi_k(t) = \sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r t^n = \frac{t^k}{\prod_{j=0}^{k} (1-t(j+r))}. \blacksquare$$

An important implication of the generating function in Theorem 13 is the subsequent formula represented in symmetric function form.

Theorem 14. *The explicit formula in homogeneous function form for the second kind r-Stirling numbers is provided for nonnegative integers n and k:*

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{s_0+s_1+s_2+\cdots+s_k=n-k} \prod_{j=0}^{k} (j+r)^{s_j}.$$

Equivalently, we have

$$\left\{ {n+m \atop n} \right\}_r = \sum_{r\leq j_1\leq j_2\leq \cdots \leq j_m \leq n} \prod_{i=1}^{m} j_i.$$

Proof: The generating function as described in Theorem 13 can be written as

$$\sum_{n\geq k} \left\{ {n+r \atop k+r} \right\}_r t^n = \frac{t^k}{\prod_{j=0}^{k} (1-t(j+r))}$$

$$= t^k \prod_{j=0}^{k} \frac{1}{(1-t(j+r))}$$

$$= t^k \prod_{j=0}^{k} \sum_{n\geq k} \big(t(j+r)\big)^n$$

$$= t^k \prod_{j=0}^{k} \sum_{n\geq k} (j+r)^n t^n$$

$$= t^k \sum_{n\geq k} \sum_{s_0+s_1+s_2+\cdots+s_k=n-k} \prod_{j=0}^{k} (j+r)^{s_j} t^{s_j}$$

$$= t^k \sum_{n\geq k} \sum_{s_0+s_1+s_2+\cdots+s_k=n-k} t^{n-k} \prod_{j=0}^{k} (j+r)^{s_j}$$

$$= \sum_{n\geq k} \left\{ \sum_{s_0+s_1+s_2+\cdots+s_k=n-k} \prod_{j=0}^{k} (j+r)^{s_j} \right\} t^n.$$

Hence, comparing the coefficients of t^n, we have

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{s_0+s_1+s_2+\cdots+s_k=n-k} \prod_{j=0}^{k} (j+r)^{s_j}.$$

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{s_0+\cdots+s_k=n-k} (0+r)\ldots(0+r) \cdot (1+r)\ldots(1+r)\cdot \ldots \cdot (k+r)\ldots(k+r)$$

$$\left\{ {n+r \atop k+r} \right\}_r = \sum_{0\le j_1\le j_2\le\cdots\le j_{n-k}\le k} \prod_{i=1}^{n-k} (j_i + r).$$

Replacing n with $n+m-r$ and k with $n-r$, we get

$$\left\{ {n+m-r+r \atop n-r+r} \right\}_r = \sum_{0\le j_1\le\cdots\le j_{n+m-r-(n-r)}\le n-r} \prod_{i=1}^{(n+m-r)-(n-r)} (j_i + r)$$

$$\left\{ {n+m \atop n} \right\}_r = \sum_{0\le j_1\le j_2\le\cdots\le j_m\le n-r} \prod_{i=1}^{m} (j_i + r) \quad (9)$$

$$\left\{ {n+m \atop n} \right\}_r = \sum_{r\le j_1\le j_2\le\cdots\le j_m\le n+r-r} \prod_{i=1}^{m} (j_i - r + r)$$

$$\left\{ {n+m \atop n} \right\}_r = \sum_{r\le j_1\le j_2\le\cdots\le j_m\le n} \prod_{i=1}^{m} j_i . \blacksquare.$$

The significance of Theorem 14 lies in linking the second kind r-Stirling numbers with the notions of 0-1 tableau and A-tableau. As articulated in (De Medicis and Leroux, 1995), an A-tableau, denoted by φ, corresponds to a sequence of column c within a Ferrer's diagram of a partition λ. These columns are organized in a descending order of length, and the lengths |c| are chosen from the sequence $A = (a_i)_{i\ge 0}$, where A represents a strictly ascending sequence of non-negative integers, as defined in (De Medicis and Leroux, 1995).

It is important to observe that an A-tableau can be created by specifying the count of columns whose lengths belong to the sequence A. For instance, considering $A = \{1,2,3,4\}$, the A-tableaux characterized by precisely 3 columns, with their lengths being part of the sequence A, can be expressed in terms of multisets. In this representation, the entries consist of column lengths, rather than entire columns, and can be articulated as follows: {4,4,4} {4,4,3} {4,4,2} {4,4,1} {4,3,3} {4,3,2} {4,3,1} {4,2,2} {4,2,1} {4,1,1} {3,3,3} {3,3,2} {3,3,1} {3,2,2} {3,2,1} {3,1,1} {2,2,2} {2,2,1} {2,1,1} {1,1,1}.

This implies that the number of such A-tableaux is the same as the number of 3-element multi-subsets of the multiset $\{\infty \cdot 1, \infty \cdot 2, \infty \cdot 3, \infty \cdot 4\}$ which is given by $H_3^4 = 20$ (Chen and Koh, 1992). In general, the number of r-element multi-subsets of a multiset $M = \{\infty \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_n\}$ as given in (Chen and Koh, 1992) is

$$H_r^n = \binom{r+n-1}{r}.$$

Therefore, if $T^A(k,r)$ represents the collection of A-tableaux featuring r columns, where the lengths − which may not necessarily be distinct − belong to the set $\{0,1,2,\ldots,k\}$, then

$$|T^A(k,r)| = \binom{r+k}{r}.$$

Consider a function ω: N^* to K, where N^* is the set of non-negative integers and K is a ring. Assuming $\mathbf{\Phi}$ represents an A-tableau featuring r columns whose lengths $|c|$ are less than or equal to h, we define

$$\omega(\mathbf{\Phi}) = \prod_{c\in\mathbf{\Phi}} \omega(|c|).$$

It is worth noting that $\mathbf{\Phi}$ could include a finite number of columns with zero lengths, given that 0 belongs to $A = \{0,1,2,\ldots,k\}$, and assuming $\omega(0) \neq 0$. Moving forward, whenever an A-tableau is referenced, it is invariably linked to the sequence $A = \{0,1,2,\ldots,k\}$.

Theorem 15. *Consider $\omega : N^* \to K$ as the column weight based on length, defined by $\omega(|c|) = |c| + r$, where $|c|$ represents the column length in A-tableau within the set $T^A(k, n-k)$. Then,*

$$\left\{ {n+k \atop n} \right\}_r = \sum_{\phi\in T^A(k,n-k)} \prod_{c\in\phi} \omega(|c|).$$

Proof. This immediately follows from (9).

By converting A-tableau columns in $T^A(k,n-k)$ into column lengths $\omega(|c|)$, we create a tableau known as an A_ω-tableau. It is important to note that when $\omega(|c|)$ equals $|c|$, the A_ω-tableau is essentially the A-tableau. Here, we introduce an $A_\omega(0,1)$-tableau as a 0-1 tableau derived by filling the cells of an A_ω-tableau using 0 and 1 in a manner where only one 1 appears in each column. The set of all such

$A_\omega(0,1)$-tableaux is denoted as $T^{A_\omega(0,1)}(k, n-k)$. Hence, the second kind r-Stirling numbers may be interpreted as follows:

$\left\{ n+k \atop n \right\}_r$ = *the number of possible* $A_\omega(0,1)$-*tableaux in* $T^{A_\omega(0,1)}(k, n-k)$ *where* $\omega(|c|) = |c| + r$.

Through the utilization of Theorem 2 and the definition of Aω(0,1)-tableau, we derive the following corollaries.

For the formula of the signed first kind r-Stirling numbers in symmetric function form, we have the subsequent theorem.

Theorem 16. *The signed first kind r-Stirling numbers equal*

$$\widehat{\left[n \atop n-m \right]}_r = (-1)^{n-k} \sum_{r \le j_1 < j_2 < \cdots < j_m < n} j_1 j_2 \dots j_m.$$

Equivalently,

$$\left[n \atop n-m \right]_r = \sum_{r \le j_1 < j_2 < \cdots < j_m < n} j_1 j_2 \dots j_m.$$

Proof: Taking the derivative with respect to z to both sides of the following horizontal generating function

$$\sum_{k=0}^{n} \widehat{\left[n+r \atop k+r \right]}_r z^k = (z-r)(z-r-1)\dots(z-r-(n-1)),$$

we have

$$\frac{d}{dz}\Big((z-r)(z-r-1)\dots(z-r-(n-1))\Big) = \sum_{k=0}^{n} \widehat{\left[n+r \atop k+r \right]}_r \frac{d}{dz} z^k$$

$$\sum_{0 \le j_1 < j_2 < \cdots < j_{n-1} \le n-1} (z-r-j_1)(z-r-j_2)\dots(z-r-j_{n-1}) = \sum_{k=1}^{n} \widehat{\left[n+r \atop k+r \right]}_r k(z)^{k-1}.$$

Taking the second derivative gives

$$\sum_{0 \le j_1 < j_2 < \cdots < j_{n-1} \le n-1} \frac{d}{dz}(z-r-j_1)(z-r-j_2)\dots(z-r-j_{n-1}) = \sum_{k=1}^{n} \widehat{\left[n+r \atop k+r \right]}_r k \frac{d}{dz}(z^{k-1})$$

$$\sum_{0 \le j_1 < j_2 < \cdots < j_{n-2} \le n-1} 2(z-r-j_1)(z-r-j_2)\dots(z-r-j_{n-2}) = \sum_{k=1}^{n} \widehat{\left[n+r \atop k+r \right]}_r k(k-1)z^{k-2}$$

$$2 \sum_{0 \le j_1 < j_2 < \cdots < j_{n-2} \le n-1} (z-r-j_1)(z-r-j_2)\dots(z-r-j_{n-2}) = \sum_{k=1}^{n} \widehat{\left[n+r \atop k+r \right]}_r k(k-1)z^{k-2}.$$

Applying 3rd derivative, we have

$$2 \sum_{0 \le j_1 < j_2 < \cdots < j_{n-3} \le n-1} 3(z-r-j_1)(z-r-j_2)\dots(z-r-j_{n-3}) = \sum_{k=1}^{n} \widehat{\left[n+r \atop k+r \right]}_r k(k-1)(k-2)z^{k-3}.$$

By induction, the k^{th} derivative evaluated at $z = 0$ gives

$$\frac{d^k}{dz^k}\Big((z-r)(z-r-1)\dots(z-r-(n-1))\Big)\Big|_{z=0} = \sum_{k=0}^{n} \widehat{\left[n+r \atop k+r \right]}_r \frac{d^k}{dz^k} z^k \Big|_{z=0}$$

$$k! \sum_{0 \le j_1 < j_2 < \cdots < j_{n-k} \le n-1} (0-r-j_1)(0-r-j_2)\dots(0-r-j_{n-k}) = \widehat{\left[n+r \atop k+r \right]}_r k!$$

$$\sum_{0 \le j_1 < j_2 < \cdots < j_{n-k} \le n-1} (0-r-j_1)(0-r-j_2)\dots(0-r-j_{n-k}) = \widehat{\left[n+r \atop k+r \right]}_r$$

$$\sum_{0 \le j_1 < j_2 < \cdots < j_{n-k} \le n-1} (-1)^{n-k}(r+j_1)(r+j_2)\dots(r+j_{n-k}) = \widehat{\left[n+r \atop k+r \right]}_r$$

$$(-1)^{n-k} \sum_{0 \le j_1 < j_2 < \cdots < j_{n-k} \le n-1} (r+j_1)(r+j_2)\dots(r+j_{n-k}) = \widehat{\left[n+r \atop k+r \right]}_r$$

$$(-1)^{n-k} \sum_{r \le j_1 < j_2 < \cdots < j_{n-k} \le n+r-1} j_1 j_2 \dots j_{n-k} = \widehat{\left[n+r \atop k+r \right]}_r$$

$$\sum_{r \le j_1 < \cdots < j_{n-r-(n-m-r)} \le n-r+r-1} j_1 \dots j_{n-r-(n-m-r)} = (-1)^{n-r-(n-m-r)} \widehat{\left[n-r+r \atop n-m-r+r \right]}_r$$

$$\sum_{r\le j_1<j_2<\cdots<j_m<n} j_1j_2\dots j_m = (-1)^{n-(n-m)}\widehat{\left[\begin{matrix} n \\ n-m \end{matrix}\right]}_r$$

$$(-1)^{n-(n-m)}\widehat{\left[\begin{matrix} n \\ n-m \end{matrix}\right]}_r = \sum_{r\le j_1<j_2<\cdots<j_m<n} j_1j_2\dots j_m.$$

Equivalently,

$$\left[\begin{matrix} n \\ n-m \end{matrix}\right]_r = \sum_{r\le j_1<j_2<\cdots<j_m<n} j_1j_2\dots j_m . \blacksquare$$

Horizontal and Vertical Recurrence Relations

Here, we establish other forms of recursive relations: the horizontal and vertical recursive relations. These relations are analogous to the hockey stick identities of the binomial coefficients.

Theorem 17. *The horizontal recursive relation satisfied by the second kind r-Stirling number is as follows:*

$$\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r = \sum_{j=0}^{n-k}(-1)^j\,(k+r+1)^{\bar{j}}\left\{\begin{matrix} n+r+1 \\ k+r+j+1 \end{matrix}\right\}_r.$$

Proof. By making use of the triangular recurrence relation of the r-Stirlinng numbers of the second kind, we have

$$\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r =$$

$$= \sum_{j=0}^{n-k}(-1)^{j+1}(k+r+1)^{\overline{j+1}}\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r$$

$$+\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r + \sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\overline{j+1}}\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r$$

$$= \sum_{j=-1}^{n-k}(-1)^{j+1}(k+r+1)^{\overline{j+1}}\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r$$

$$+\sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\overline{j+1}}\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r$$

$$= \sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\bar{j}}\left\{\begin{matrix} n+r \\ k+r+j \end{matrix}\right\}_r$$

$$+\sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\bar{j}}(k+r+1+j)\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r$$

$$= \sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\bar{j}}\left\{\left\{\begin{matrix} n+r \\ k+r+j \end{matrix}\right\}_r + (k+r+1+j)\left\{\begin{matrix} n+r \\ k+r+j+1 \end{matrix}\right\}_r\right\}$$

$$= \sum_{j=0}^{n-k}(-1)^j(k+r+1)^{\bar{j}}\left\{\begin{matrix} n+r+1 \\ k+r+j+1 \end{matrix}\right\}_r . \blacksquare$$

The following theorem contains another form of a recurrence relation, which is a consequence of the above rational generating function.

Theorem 18. *The vertical recursive relation adhered to by the second kind r-Stirling numbers is as follows:*

$$\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r = \sum_{j=0}^{n}(k+r)^{n-j}\left\{\begin{matrix} j+r-1 \\ k+r-1 \end{matrix}\right\}_r .$$

Proof. Using Theorem 13, we have

$$\sum_{n=0}^{\infty}\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r t^n = \frac{t^k}{\prod_{j=0}^{k}(1-t(j+r))}$$

$$= \frac{t}{1-t(k+r)}\frac{t^{k-1}}{\prod_{j=0}^{k-1}(1-t(j+r))}$$

$$= \left(t\sum_{n=0}^{\infty}\big(t(k+r)\big)^n\right)\left(\sum_{n=k}^{\infty}\left\{\begin{matrix} n+r-1 \\ k+r-1 \end{matrix}\right\}_r t^{n-1}\right)$$

$$= \left(\sum_{n=0}^{\infty}\big(t(k+r)\big)^n\right)\left(\sum_{n=k}^{\infty}\left\{\begin{matrix} n+r-1 \\ k+r-1 \end{matrix}\right\}_r t^{n}\right)$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}\big(t(k+r)\big)^{n-j}\left\{\begin{matrix} j+r-1 \\ k+r-1 \end{matrix}\right\}_r t^j\right\}$$

$$= \sum_{n=0}^{\infty}\left\{\sum_{j=0}^{n}(k+r)^{n-j}\left\{\begin{matrix} j+r-1 \\ k+r-1 \end{matrix}\right\}_r\right\}t^n .$$

By comparing coefficient of t^n both sides, we have

$$\left\{\begin{matrix} n+r \\ k+r \end{matrix}\right\}_r = \sum_{j=0}^{n}(k+r)^{n-j}\left\{\begin{matrix} j+r-1 \\ k+r-1 \end{matrix}\right\}_r . \blacksquare$$

It is worth-mentioning that the recurrence relations in Theorems 15 and 16 are not considered by Broder (1984).

CONCLUSION AND RECOMMENDATION

A slightly modified r-Stirling numbers, also called the SM r-Stirling numbers, were successfully defined by means of exponential generating function. An algebraic approach has been employed to uncover interesting properties and identities, encompassing the exploration of horizontal generating functions, orthogonality and inverse relations, triangular recurrence relations, explicit formulas, rational generating functions, and explicit formulas in symmetric function form. A new combinatorial interpretation for second kind r-Stirling numbers has been constructed in terms the combinatorics of 0-1 tableau using the explicit formulas in Theorem 14 in symmetric function form. The paper has been concluded by introducing two innovative structures of recursive relations: the horizontal and vertical recursive formulas, analogous to the famous "Hockey Stick" identity for binomial coefficients.

The method employed in defining the SM r-Stirling numbers shares similarities with the conventional approach used for defining Bernoulli, Euler, and Genocchi numbers (Abramowitz and Stegun, 1970), (Agoh, 2014), (Araci, 2012), (Araci, 2014), (Corcino, 2020), (Kim et al., 2012). Consequently, there is a potential to introduce novel variants of SM r-Stirling numbers by integrating them with the concepts of Bernoulli, Euler, and Genocchi numbers. Specifically, these variants may be defined as follows:

$$\sum_{n=0}^{\infty} SB_n^1(k;r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{t\ln^k(1+t)}{k!\,(e^t-1)}$$

$$\sum_{n=0}^{\infty} SB_n^2(k;r)\frac{t^n}{n!} = \frac{te^{rt}(e^t-1)^{k-1}}{k!}$$

$$\sum_{n=0}^{\infty} SE_n^1(k;r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{2\ln^k(1+t)}{k!\,(e^t+1)}$$

$$\sum_{n=0}^{\infty} SE_n^2(k;r)\frac{t^n}{n!} = \frac{2te^{rt}(e^t-1)^k}{k!\,(e^t+1)}$$

$$\sum_{n=0}^{\infty} SG_n^1(k;r)\frac{t^n}{n!} = \left(\frac{1}{1+t}\right)^r \frac{2t\ln^k(1+t)}{k!\,(e^t+1)}$$

$$\sum_{n=0}^{\infty} SG_n^2(k;r)\frac{t^n}{n!} = \frac{2te^{rt}(e^t-1)^k}{k!\,(e^t+1)}$$

where $SB_n^1(k;r)$, $SB_n^2(k;r)$, $SE_n^1(k;r)$, $SE_n^2(k;r)$, $SG_n^1(k;r)$ and $SG_n^2(k;r)$ may denote the first and second kinds r-Stirling-Bernoulli numbers, the first and second kinds r-Stirling-Euler numbers and the first and second kinds r-Stirling-Genocchi numbers, respectively.

It is also interesting to establish a q-analogue of SM r-Stirling numbers following the method used by Corcino and Montero (2012) in establishing the q-analogue of Rucinski-Voigt numbers. Further insights can be gleaned from the works of Corcino and Barrientos (2011) and Corcino and Corcino (2012).

ACKNOWLEDGEMENT

The authors would like to extend their appreciation to the referees for their corrections and suggestions, which have contributed to enhancing the clarity of this paper.

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