CNU Journal of Higher Education
Volume 14 (2020), p 1-6

The Cubic Lambert W Function

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Date Submitted: December 2019
Date Revised: June 20, 2020
Originality: 85%
Plagiarism Detection: Passed

ABSTRACT

This paper presents a cubic generalization of the Lambert W function, or simply called the Cubic Lambert W function, and establish its Taylor series, derivative and its real branches, Taylor series expansion of its integral and its Mellin transform. The Mellin transform is known to be useful in computer science in the analysis of algorithm and in number theory in the analysis of the prime-counting function while it is a fact that Taylor series expansion of the function is useful to obtain approximation values. In the final section of the paper, the derivative and the real branches of the function are used to establish that the fixed-point solution in the application to a nonlinear differential equation is stable.
Keywords: *Lambert function, Taylor series, Mellin transform, real branches*

INTRODUCTION

The Lambert W function which is the solution to the equation $we^w = x$ appears in many applications such as in a solution of a jet fuel problem, of a model combustion problem, of an enzyme kinetics problem, of linear constant-coefficient delay equations, and has combinatorial applications and many others (Corless et al., 1996). The Lambert W function was named after the Swiss polymath Johann Heinrich Lambert (1728-1777), who first introduced the function in 1758. The letter W for this function is due to an early Maple usage. An additional significance of the letter W in the function is the pioneering work on many aspects of W by Wright (1959).

A motivation to generalize the Lambert W function is the asymptotic estimation of the Bell and generalized Bell numbers (Mezö and Baricz, 2017). The application to generalized Bell numbers is found in finding the asymptotic behavior of the r-Bell numbers as $n \to \infty$ and r is fixed (Corcino and Corcino, 2013).

A quadratic generalization of the Lambert W function has been studied separately. In the present paper a cubic generalization is presented and we derived its basic analytic properties such as derivative, integral and Taylor series, Mellin transform and its real branches for a positive parameter are discussed.

BASIC PROPERTIES OF CUBIC LAMBERT FUNCTION

Derivative of the Cubic Lambert and Its Real Branches

Consider the following equation that defines the cubic Lambert function $y = W(3; a, x)$

$$ye^{ay^3+y} = x, \quad a \neq 0. \qquad (2.1)$$

Let $y = W(3; a, x) := W$. Then taking the implicit differentiation with respect to x, we have

$$\frac{d}{dx}\left(We^{aW^3+W} = x\right)$$

$$We^{aW^3+W}(3aW^2 \cdot W' + W') + W' \cdot e^{aW^3+W} = 1$$

$$\left(We^{aW^3+W}(3aW^2+1) + e^{aW^3+W}\right)W' = 1$$

$$W'(3; a, x) = \frac{e^{-aW^3-W}}{W(3aW^2+1)+1} = \frac{e^{-aW^3-W}}{3aW^3+W+1}.$$

This result is formally stated in the following theorem.

Theorem 2.1. The cubic Lambert function $y = W(3; a, x)$ has the following derivative

$$W'(3;a,x) = \frac{e^{-aW^3-W}}{3aW^3+W+1}.$$

The singularities of $W'(3;a,x)$ are the zeros of the denominator

$$3aW^3 + W + 1 = 0, \qquad (2.2)$$

which are given by $\{u_1, u_2, u_3\}$ where

$$u_1 = \frac{1}{3}\left\{\frac{-2^{\frac{1}{3}}}{\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}} + \frac{\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}}{2^{\frac{1}{3}}a}\right\};$$

$$u_2 = \frac{1+i\sqrt{3}}{3\left(2^{\frac{2}{3}}\right)\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}} - \frac{\left(1-i\sqrt{3}\right)\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}}{6\left(2^{\frac{1}{3}}\right)a};$$

$$u_3 = \frac{1-i\sqrt{3}}{3\left(2^{\frac{2}{3}}\right)\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}} - \frac{\left(1+i\sqrt{3}\right)\left(-9a^2+\sqrt{4a^3+81a^4}\right)^{\frac{1}{3}}}{6\left(2^{\frac{1}{3}}\right)a}.$$

The real root u_1 is essential in determining the real branches of the cubic Lambert W function. We shall assume here that $a > 0$. Note that when $x = 0$, $W(3; a, 0) = 0$ and as $x \to \infty$, $y \to \infty$.

It can be verified using mathematica that

$$u_1 < 0 \iff -\frac{4}{81} \leq a < 0 \ \& \ a > 0.$$

Now, equation (2.2) can be written as

$$\begin{aligned}(w-u_1)(w-u_2)(w-u_3) &= 0\\ &= (w-u_1)(w-u_2)(w-\overline{u_2})\\ &= (w-u_1)(w^2 - w(u_2+\overline{u_2}) + u_2\overline{u_2}),\end{aligned}$$

since $u_3 = \bar{u}_2$. Thus, the function $W(3; a, x)$ is differentiable throughout $\mathbb{R}$ except at $W = u_1$, that is at $x = f_a(u_1)$, where

$$f_a(u_1) = u_1 e^{au_1^3+u_1}.$$

The function has only two real branches which are separated by the line $W = u_1$. Note that $W(3; a, 0) = 0$ and as $x \to \infty$, $W(3; a, x) \to \infty$. Also, if < 0, $W(3; a, x) < 0$. Thus, $W(3; a, x)$ must be a strictly increasing function for $W > u_1$ and $u_1 < 0$.

Write $3aW^3 + W + 1 = (W - u_1)P$, where $P = W^2 - W(u_2 + \overline{u_2}) + u_2\overline{u_2}$. P is always positive as verified using mathematica.

If $W < u_1$, then $(W - u_1) < 0$. Since $P > 0$, the derivative $W'(3; a, x) < 0$. Thus, $W(3; a, x)$ is strictly decreasing for $W < u_1$.

Therefore, the two branches of $W(3; a, 0)$ for $a > 0$ are:

$W_0(3; a, 0)\colon (f_a(u_1), +\infty) \longrightarrow (u_1, +\infty)$ which is a strictly increasing function, differentiable in the interior of its domain;

$$W_1(3;a,0)\colon \left(-\infty, f_a(u_1)\right) \longrightarrow (-\infty, u_1)$$

which is a strictly decreasing function, differentiable in the interior of its domain.

Deriving the Taylor Series Expansion of Cubic Lambert Function

Given the cubic Lambert function $y = W(3; a, x)$, its inverse function is given by

$$x = f(W) = We^{aW^3+W}$$

By Lagrange Inversion Formula, we have

$$\begin{aligned}W(3;a,0) &= g(x)\\ &= a + \sum_{n=1}^{\infty} g_n \frac{\left(x - f(a)\right)^n}{n!}\end{aligned}$$

where

$$g_n = \lim_{w\to a}\left[\frac{d^{n-1}}{dw^{n-1}}\left(\frac{w-a}{f(w)-f(a)}\right)\right]^n.$$

Note that when $a=0$,

$$f(0) = 0e^{a(0^3)+0} = 0.$$

Hence,

$$W(3;a,0) = g(x) = \sum_{n=1}^{\infty}\frac{x^n}{n!}\lim_{w\to 0}\frac{d^{n-1}}{dw^{n-1}}\left(\frac{w}{we^{aw^3+w}}\right)^n.$$

Therefore,

$$W(3;a,x) = \sum_{k=1}^{\infty}\frac{x^k}{k!}\lim_{w\to 0}\frac{d^{k-1}}{dw^{k-1}}\left(\frac{w}{we^{aw^3+w}}\right)^k$$

$$\frac{d^{k-1}}{dw^{k-1}}\left(\frac{w}{we^{aw^3+w}}\right)^k = \frac{d^{k-1}}{dw^{k-1}}e^{-k(aw^3+w)} = \frac{d^{k-1}}{dw^{k-1}}\sum_{m=0}^{\infty}\frac{(-k)^m(aw^3+w)^m}{m!}$$

$$= \sum_{m=0}^{\infty}\frac{(-k)^m}{m!}\frac{d^{k-1}}{dw^{k-1}}(aw^3+w)^m = \sum_{m=0}^{\infty}\frac{(-k)^m}{m!}\sum_{i=0}^{m}\binom{m}{i}\frac{d^{k-1}}{dw^{k-1}}(aw^3)^i w^{m-i}.$$

$$\left.\frac{d^{k-1}}{dw^{k-1}}\left(\frac{w}{we^{aw^3+w}}\right)\right|_{w=0}$$

$$= \left.\sum_{m=0}^{\infty}\frac{(-k)^m}{m!}\sum_{i=0}^{m}\binom{m}{i}a^i\frac{d^{k-1}}{dw^{k-1}}w^{3i+m-i}\right|_{w=0} = \left.\sum_{m=0}^{\infty}\sum_{i=0}^{m}\frac{(-k)^m}{m!}\binom{m}{i}a^i\frac{d^{k-1}}{dw^{k-1}}w^{2i+m}\right|_{w=0}$$

$$= \left.\sum_{m=0}^{\infty}\sum_{i=0}^{m}\frac{(-k)^m}{m!}\binom{m}{i}a^i(2i+m)_{k-1}w^{2i+m-(k-1)}\right|_{w=0}$$

$$= \sum_{m=\left\lfloor\frac{k-1}{3}\right\rfloor}^{k-1}\frac{(-k)^m}{m!}\binom{m}{\frac{k-m-1}{2}}a^{\frac{k-m-1}{2}}\times(k-1)_{k-1}$$

$$= \sum_{m=\left\lfloor\frac{k-1}{3}\right\rfloor}^{k-1}\frac{(-k)^m}{m!}\binom{m}{\frac{k-m-1}{2}}a^{\frac{k-m-1}{2}}\times(k-1)!$$

$$W(3;a,x) = \sum_{k=1}^{\infty}\frac{x^k}{k!}\sum_{m=\left\lfloor\frac{k-1}{3}\right\rfloor}^{k-1}\frac{(-k)^m}{m!}\binom{m}{\frac{k-m-1}{2}}a^{\frac{k-m-1}{2}}\times(k-1)!$$

$$W(3;a,x) = \frac{1}{\sqrt{a}}\sum_{k=1}^{\infty}\frac{(\sqrt{a}x)^k}{k}\sum_{m=\left\lfloor\frac{k-1}{3}\right\rfloor}^{k-1}\frac{(-k/\sqrt{a})^m}{m!}\binom{m}{\frac{k-m-1}{2}}$$

We formally state the theorem as follows.

Theorem 2.2. The Taylor series of the cubic Lambert W function around $x=0$ is given by

$$W(3;a,x) = \frac{1}{\sqrt{a}}\sum_{k=1}^{\infty}\frac{(\sqrt{a}x)^k}{k}\sum_{m=\left\lfloor\frac{k-1}{3}\right\rfloor}^{k-1}\frac{(-k/\sqrt{a})^m}{m!}\binom{m}{\frac{k-m-1}{2}}.$$

Integral of Cubic Lambert Function

Taking the derivative of both sides of equation (2.1) gives

$$dx = ye^{ay^3+y}(3ay^2+1)\,dy + e^{ay^3+y}\,dy = e^{ay^3+y}(3ay^3+y+1)\,dy$$

With $y = W(3;a,x)$, we have

$$\int W(3;a,x)dx = \int y\,dx$$
$$= \int y\left(e^{ay^3+y}\right)(3ay^3+y+1)\,dy$$
$$= \int (3ay^4+y^2+y)e^{ay^3+y}\,dy$$
$$= \int (3ay^4+y^2)e^{ay^3+y}\,dy + \int ye^{ay^3+y}\,dy$$
$$= \int y^2e^{ay^3+y}(3ay^2+1)\,dy + \int ye^{ay^3+y}\,dy = I_1 + I_2$$

Solving I_1 using integration by parts, with $u=y^2$ and

$$dv = e^{ay^3+y}(3ay^2+1)\,dy.$$

Then, we have $du = 2y dy$ and $v = e^{ay^3+y}$. Hence,

$$I_1 = uv - \int v\, du$$
$$= y^2 e^{ay^3+y} - \int 2y e^{ay^3+y} dy + C$$

Thus, we obtain

$$\int W(3; a, x) dx = I_1 + I_2$$
$$= y^2 e^{ay^3+y} - \int y e^{ay^3+y}\, dy$$

Using the Taylor series expansion of ye^{ay^3+y}, we can integrate term by term to find a series representation of the antiderivative which converges on the entire complex plane, since e^{ay^3+y} is an entire function. Thus, we have

$$\int y e^{ay^3+y}\, dy = \sum_{n\geq 0} \frac{1}{n!} \int y(ay^3 + y)^n dy$$
$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^k \int y^{2k+n+1} dy$$
$$= \sum_{n\geq 0} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} a^k \frac{y^{2k+n+2}}{2k+n+2} + C.$$

It follows that

$$\int W(3; a, x) dx = y^2 e^{ay^3+y}$$
$$- \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} \frac{a^k}{n!} \frac{y^{2k+n+2}}{2k+n+2} + C.$$

Then, we have

Theorem 2.3. The integral of cubic Lambert W function has the following Taylor series expansion

$$\int W(3; a, x)\, dx$$
$$= W^2(3; a, x) e^{aW^3(3;a,x)+W(3;a,x)}$$
$$- \sum_{n\geq 0} \sum_{k=0}^{n} \binom{n}{k} \frac{a^k}{n!} \frac{[W(3; a, x)]^{2k+n+2}}{2k+n+2} + C.$$

THE MELLIN TRANSFORMATION OF CUBIC LAMBERT FUNCTION

The Mellin transformation is one of the special cases of integral transformations.

If $W(3; a, x)$ is the solution to the equation $ye^{ay^3+y} = x$, then its Mellin transform is given by

$$(\mathcal{M}W(3; a, x))(s)$$
$$= \int_0^\infty t^{s-1} W(3; a, t)\, dt.$$

Letting $u = W(3; a, t)$. Then, we have

$$ue^{au^3+u} = t.$$

Applying implicit differentiation, yields

$$ue^{au^3+u}(3au^2 + 1)\, du + du\, e^{au^3+u} = dt$$

$$e^{au^3+u}(3au^3 + u + 1) du = dt.$$

Thus,

$$(\mathcal{M}W(3; a, x))(s)$$

$$= \int_0^\infty \left(ue^{au^3+u}\right)^{s-1} u\, e^{au^3+u}(3au^3 + u + 1)\, du$$
$$= \int_0^\infty \left(ue^{au^3+u}\right)^{s} (3au^3 + u + 1)\, du$$
$$= \int_0^\infty u^s e^{s(au^3+u)} (3au^3 + u + 1)\, du$$

$$= \int_0^\infty \{3au^{s+3} e^{s(au^3+u)} + u^{s+1} e^{s(au^3+u)} + u^s e^{s(au^3+u)}\} du$$
$$= 3a \int_0^\infty u^{s+3} e^{s(au^3+u)}\, du$$
$$+ \int_0^\infty u^{s+1} e^{s(au^3+u)}\, du + \int_0^\infty u^s e^{s(au^3+u)}\, du.$$

Thus, the Mellin transform of $W(3;a,x)$ is

$$\begin{aligned}(\mathcal{M}W(3;a,x))(s) &= 3a\left\{\frac{1}{6}\, s(-as)^{\frac{-8-s}{3}}\left[-2a(-as)^{\frac{1}{3}}\,\mathrm{H}_1 + 2as\,H_2 + s(-as)^{\frac{2}{3}}\,H_3\right]\right\}\\ &+\frac{1}{6}\,(-as)^{\frac{-4-s}{3}}\left[2(-as)^{\frac{2}{3}}\,\mathrm{H}_4 + 2s(-as)^{\frac{1}{3}}\,H_5 + s^2\,H_6\right]\\ &+\frac{1}{6}\,(-as)^{\frac{1}{3}(-2-s)}\left[2a(-as)^{\frac{1}{3}}\,\mathrm{H}_7 + 2as\,H_8 - s(-as)^{\frac{2}{3}}\,H_9\right]\end{aligned}$$

where

$$\mathrm{H}_1 = \Gamma\left(\frac{4+\mathrm{s}}{3}\right){}_1F_2\left(\frac{4+s}{3};\frac{1}{3},\frac{2}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_2 = \Gamma\left(\frac{5+\mathrm{s}}{3}\right){}_1F_2\left(\frac{5+s}{3};\frac{2}{3},\frac{4}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_3 = \Gamma\left(\frac{6+\mathrm{s}}{3}\right){}_1F_2\left(\frac{6+s}{3};\frac{4}{3},\frac{5}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_4 = \Gamma\left(\frac{2+\mathrm{s}}{3}\right){}_1F_2\left(\frac{2+s}{3};\frac{1}{3},\frac{2}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_5 = \Gamma\left(\frac{3+\mathrm{s}}{3}\right){}_1F_2\left(\frac{3+s}{3};\frac{2}{3},\frac{4}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_6 = \Gamma\left(\frac{4+\mathrm{s}}{3}\right){}_1F_2\left(\frac{4+s}{3};\frac{4}{3},\frac{5}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_7 = \Gamma\left(\frac{1+\mathrm{s}}{3}\right){}_1F_2\left(\frac{1+s}{3};\frac{1}{3},\frac{2}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_8 = \Gamma\left(\frac{2+\mathrm{s}}{3}\right){}_1F_2\left(\frac{2+s}{3};\frac{2}{3},\frac{4}{3};\frac{-s^2}{27a}\right)$$
$$\mathrm{H}_9 = \Gamma\left(\frac{3+\mathrm{s}}{3}\right){}_1F_2\left(\frac{3+s}{3};\frac{4}{3},\frac{5}{3};\frac{-s^2}{27a}\right)$$

$s > 1$ and

$$pFq\left(a_1,\dots,a_p;b_1,\dots,b_q;z\right) = 1+\frac{a_1\cdots a_p}{b_1\cdots b_q}\frac{z}{1!}+\frac{a_1(a_1+1)\cdots a_p(a_p+1)}{b_1(b_1+1)\cdots b_q(b_q+1)}\frac{z^2}{2!}+\cdots$$

is the generalized hypergeometric function.

Application to Nonlinear Differential Equations

Consider the nonlinear differential equation

$$T\frac{dx}{dt} = b - xe^{ax^3+x}, \qquad (4.1)$$

where T represents temperature, a, b are constants, $b \neq 0$. We shall solve (4.1) by perturbation of the solution from the fixed point x_*. Let

$$f(x) = -\frac{1}{T}xe^{ax^3+x}+\frac{b}{T}\,. \qquad (4.2)$$

Then (4.1) can be written

$$\frac{dx}{dt} = f(x). \qquad (4.3)$$

We say that x_* is a fixed point, or equilibrium point of (4.3) if $f(x_*) = 0$ (Chasnov, 2019). Solving for x_*, set $f(x) = 0$. Then

$$-\frac{1}{T}xe^{ax^3+x}+\frac{b}{T} = 0$$
$$xe^{ax^3+x} = b.$$

That is,

$$x = W(3;a,b) = x_*. \qquad (4.4)$$

The perturbation method can be done as follows: Let

$$x = x_* + \varepsilon(t). \qquad (4.5)$$

Because x_* is constant,

$$\frac{dx}{dt} = \frac{d\varepsilon(t)}{dt}.$$

We use the notation

$$\dot{x} = \frac{dx}{dt}.$$

Then, (4.5) becomes $\dot{x} = \dot{\varepsilon}$. Taylor series expansion about $\varepsilon = 0$ yields

$$\begin{aligned}\dot{\varepsilon} &= f(x_* + \varepsilon)\\ &= f(x_*) + \varepsilon f'(x_*) + \cdots\\ &= \varepsilon f'(x_*) + \cdots\end{aligned}$$

The omitted term in the Taylor Series expansion is $O(\varepsilon^2)$ and can be negligible by taking $\varepsilon(0)$ sufficiently small. Therefore, at least over short times the differential equation to be considered is

$$\dot{\varepsilon} = f'(x_*)\varepsilon$$

which has the familiar solution

$$\varepsilon(t) = \varepsilon(0)e^{f'(x_*)t}.$$

We have seen in (4.4) that we only have one fixed point for specific values of a, b. Thus,

$$x = x_* + \varepsilon(0)e^{f'(x_*)t}.$$

The derivative $f'(x_*)$ is given by

$$f'(x_*) = -\frac{b}{T}\left(3a[W(3;a,b)]^2 + \frac{1}{W(3;a,b)}\right).$$

Using (4.5), $\varepsilon(0) = x_0 - x_*$, consequently,

$$x = W(3;a,b) + \left(x_0 - W(3;a,b)\right)E$$

where

$$E = \exp\left(-\frac{bt}{T}\left(3a[W(3;a,b)]^2 + \frac{1}{W(3;a,b)}\right)\right).$$

Note that for $b > 0, W(3;a,b) > 0$ as can be seen in the discussion of the branches of the function. Thus, for $T > 0$ and $b > 0, f'(x_*) < 0$ which means that the fixed point is stable.

ACKNOWLEDGEMENT

This research is funded by Cebu Normal University and the Commission on Higher Education-Grants-in-Aid for Research. The authors would like to thank the two referees for reading and evaluating the paper thoroughly, which resulted to a more improved version of the paper.

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